Math 250 [redacted]

4.3 Graphing Functions

[redacted]

Objective 1) Use calculus to identify important features, GC can deceive!

2) Sketch graph of a function showing correct
- relative extrema
- increase / decrease
- concavity
- inflection pts
- asymptotes + holes
  (vertical
  horizontal
  slant)

Types of functions you may see:

- Polynomials
- Rationals
- Rational exponents
- Radicals
- Trig functions

Use any algebra skills, too

- intercepts
- symmetry

CAUTION: The department portion of the final exam is likely to have you graph without your GC.

Math 250: Analyzing a Graph

From Algebra and Pre-calculus:

- ☐ Find x-intercepts and y-intercepts
- ☐ Check for symmetry with respect to the x-axis, the y-axis, and the origin.
- ☐ Find the domain and range.
- ☐ Identify any holes (factors that cancel out).
- ☐ Identify any vertical asymptotes.
- ☐ Identify any horizontal asymptotes.
- ☐ Identify any slant asymptotes (divide).
- ☐ Use division to identify any curvilinear asymptotes.
- ☐ Solve for values of x where the function crosses any horizontal, slant, or curvilinear asymptotes.

From Calculus:

- ☐ Check for continuity or points of discontinuity.
- ☐ Check for differentiability or points which are not differentiable.
- ☐ Use the first derivative to find intervals where the function increases and decreases.
- ☐ Use the second derivative to find intervals where the function is concave up and down.
- ☐ Use either first or second derivative tests to find relative extrema.
- ☐ Use second derivative to find points of inflection.
- ☐ Use limits at infinity to find end behavior (and confirm horizontal or slant asymptotes).

Getting it on paper:

- ☐ Place axes so that everything fits – you can move one or both axes from the center if that's helpful.
- ☐ Plot all intercepts, relative extrema, inflection points, and points of intersection with asymptotes.
- ☐ Indicate accurately where the function or its derivatives are undefined.
- ☐ Indicate all asymptotes with dashed lines.
- ☐ Make sure graph increases or decreases as indicated by your intervals.
- ☐ Make sure graph is concave up or concave down as indicated by your intervals.

To get all the credit:

- ☐ Draw and label axes, including scales if you changed from 1 box = 1 unit.
- ☐ Work neatly.
- ☐ Plot enough points to extend graph accurately to the edges of the grid.
- ☐ Label every important point (intercept, extremum, inflection) with its name or coordinates.

It is advisable to use your graphing calculator to:

- ☐ Make a table of points.
- ☐ Evaluate y-coordinates needed.
- ☐ Check shape of graph.
- ☐ Zoom in or zoom out to confirm specific details of shape.
- ☐ Identify where algebraic work may contain errors. ☹

Math 250 Using calculus to sketch important features of graphs

Analyze and sketch graph

① $f(x) = \dfrac{-2x^2 + x + 11}{x-3}$

② $y = 3(x-1)^{2/3} - (x-1)^2$

* Remember that $x^4 = 1$ has two solutions, $x = 1, -1$
so $x^{4/3} = 1$ also has two solutions.

CAUTION
Example #2!
$x = (1)^{3/4}$
$\Rightarrow x = \pm\sqrt[4]{1} = \pm 1$

③ $y = |x^2 - 6x + 5|$

Other worked examples here:

④ $f(x) = x + \dfrac{32}{x^2}$

⑤ $g(x) = x\sqrt{9-x}$

⑥ $f(x) = -x + 2\cos(x)$

Pre-calc reminder about multiplicity of roots (or zeros) and effects of odd or even multiplicity on the graph.

⑦ $f(x) = (x-2)^3(x-1)$

⑧ $y = 3x^4 - 6x^2 + \frac{5}{3}$

⑨ $p(x) = (x+2)^2(x-1)$

⑩ $p(x) = (x-1)^3(x+2)^2$

⑪ $f(x) = \dfrac{(3x+1)^2}{(x-1)^2}$

⑫ $f(x) = \dfrac{2+3x-x^3}{x}$

Extra

⑬ Find slant asymptote of $f(x) = \dfrac{5x^4 - x^3}{3x^3 + 2x^2 + 1}$

Math 250

Analyze and sketch the graph

① $f(x) = \dfrac{-2x^2 + \text{[redacted]}x + 11}{x-3}$

x ints: set $y=0$ $\qquad 0 = \dfrac{-2x^2 + \text{[redacted]}x + 11}{x-3}$

mult $(x-3)$ $\qquad 0 = -2x^2 + \text{[redacted]}x + 11$

QF $\qquad x = \dfrac{-1\text{[redacted]} \pm \sqrt{1\text{[redacted]} - 4(-2)(11)}}{2(-2)}$

$$= \frac{-1\text{[redacted]} \pm \sqrt{1\text{[redacted]} + 88}}{-4}$$

$$= \frac{-1\text{[redacted]} \pm \sqrt{\text{[redacted]}89}}{-4}$$

[redacted]

$$x = \frac{-1\text{[redacted]}}{4} + \frac{\sqrt{\text{[redacted]}89}}{4}, \quad \frac{-1\text{[redacted]}}{4} - \frac{\sqrt{\text{[redacted]}89}}{4}$$

$x \approx$ [redacted], [redacted]

y-ints: set $x=0$ $\qquad f(0) = \dfrac{-2(0)^2 + \text{[redacted]}(0) + 11}{0-3} = \dfrac{-11}{3} \approx -3.67$

domain: all $\mathbb{R}$ except denom $= 0$

$x - 3 = 0$

[redacted] $x \neq 3$. $\qquad (-\infty, 3) \cup (3, \infty)$

range: tricky! There's a gap in the middle. We'll find this after we know the relative extrema.

symmetry: $f(-x) = \dfrac{-2(-x)^2 + \text{[redacted]}(-x) + 11}{-x-3}$

$$= \frac{-2x^2 - \text{[redacted]}x + 11}{-x-3}$$

$$= \frac{-(2x^2 + \text{[redacted]}x - 11)}{-(x+3)} \neq f(x) \text{ or } -f(x)$$

no symmetry.

asymptotes: 1) no factors cancel ⇒ no holes

$x=3$ is vertical asymptote.

2) degree numerator > degree denom

→ no horizontal asymptote

Math 250

oblique asymptote: divide

$$x-3\,\overline{\big)\,-2x^2+\ \ x+11}$$

$$\begin{array}{r|rrr} 3 & -2 & -11 & 11 \\ & & -6 & -15 \\ \hline & -2 & -5 & \end{array}$$

$$-2x-5+\frac{-4}{x-3}$$

$$f(x) = \underbrace{-2x-5}_{} + \underbrace{\frac{-4}{x-3}}_{} = \frac{-2x^2+x+11}{x-3}$$

← both forms of f(x) are equivalent

↑ equation of oblique asymptote $y=-2x-5$

as $x \to \pm\infty$ this fraction $\to 0$ and has very little effect on graph.

$$f'(x) = \frac{d}{dx}\left[\frac{-2x^2+x+11}{x-3}\right]$$

quotient rule

$$= \frac{(x-3)(-4x+1)-(-2x^2+x+11)(1)}{(x-3)^2}$$

$$= \frac{-4x^2+13x-3+2x^2-x-11}{(x-3)^2}$$

$$= \frac{-2x^2+12x-14}{(x-3)^2}$$

↖ yucky way ("ugly" on next pages) to find f'(x).

or

$$\frac{d}{dx}\left[-2x-5+\frac{-4}{x-3}\right]$$

$$= \frac{d}{dx}\left[-2x-5-4(x-3)^{-1}\right]$$

$$= -2+4(x-3)^{-2}$$

$$f'(x) = -2+\frac{4}{(x-3)^2}$$

much easier to find f'(x)

↑

check: Are these equal?

$$\frac{-2(x-3)^2+4}{(x-3)^2}$$

$$= \frac{-2(x^2-6x+9)+4}{(x-3)^2}$$

$$= \frac{-2x^2+12x-18+4}{(x-3)^2}$$

$$= \frac{-2x^2+12x-14}{(x-3)^2}$$ ✓ ☺

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Find critical values by setting $f'(x) = 0$
(and noticing $f'(x)$ undefined at $x=3$, the vertical asymptote of f.)

From ugly $f'(x)$ method:

$$f'(x) = \frac{-2x^2+12x-14}{(x-3)^2}$$

numerator $= 0$

$$\frac{-2x^2}{-2}+\frac{12x}{-2}\frac{-14}{-2}=\frac{0}{-2}$$

$$x^2-6x+7=0$$

Does it factor?

$$b^2-4ac$$
$$=(-6)^2-4(1)(7)$$
$$=8$$

not a perfect square
does not factor

Quadratic formula on
$x^2-6x+7=0$

$$x = \frac{-(-6)\pm\sqrt{(-6)^2-4(1)(7)}}{2(1)}$$

$$x = \frac{6\pm\sqrt{8}}{2}$$

$$x = \frac{6}{2} \pm \frac{2\sqrt{2}}{2}$$

$x = 3\pm\sqrt{2}$ c.v.s

approx. values $x \approx 4.41, 1.59$

From easier $f'(x)$ method:

$$f'(x) = -2+\frac{4}{(x-3)^2}$$

entire function $= 0$

$$-2+\frac{4}{(x-3)^2} = 0$$

$$\frac{4}{(x-3)^2} = 2$$

$$\frac{4}{2} = (x-3)^2$$

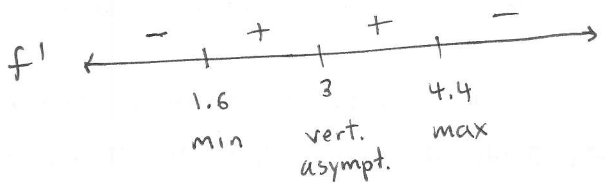
$2 = (x-3)^2$ square root property

$$\pm\sqrt{2} = x-3$$

$$3\pm\sqrt{2} = x$$

approx values $x \approx 4.41, 1.59$

Sign chart



increasing
$(3-\sqrt{2},3)\cup(3,3+\sqrt{2})$
decreasing
$(-\infty,3-\sqrt{2})\cup(3+\sqrt{2},\infty)$

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To find $f''(x)$, use $f'(x) = -2 + \frac{4}{(x-3)^2} = -2 + 4(x-3)^{2}$

Easier: $f''(x) = \frac{d}{dx}\left[-2 + 4(x-3)^{-2}\right]$

☺

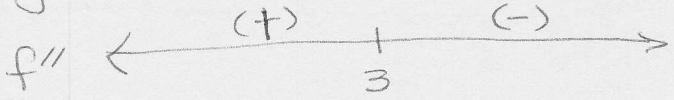
$= -8(x-3)^{-3}$

$= \frac{-8}{(x-3)^3}$

$f''(x) \neq 0$ anywhere

$f''(x)$ undef $x=3$ (but also $f'(x)$ and $f(x)$)

sign chart f'' (concavity)



no inflection points. b/c $f(3)$ not defined

concave down $(3, \infty)$

concave up $(-\infty, 3)$

Glutton for punishment? Here's the ugly way to find $f''(x)$:

$f''(x) = \frac{d}{dx}\left[\frac{-2x^2+12x-14}{(x-3)^2}\right]$ quotient rule

$= \frac{(x-3)^2(-4x+12) - (-2x^2+12x-14)\cdot 2(x-3)}{(x-3)^4}$

factor out $(x-3)$:

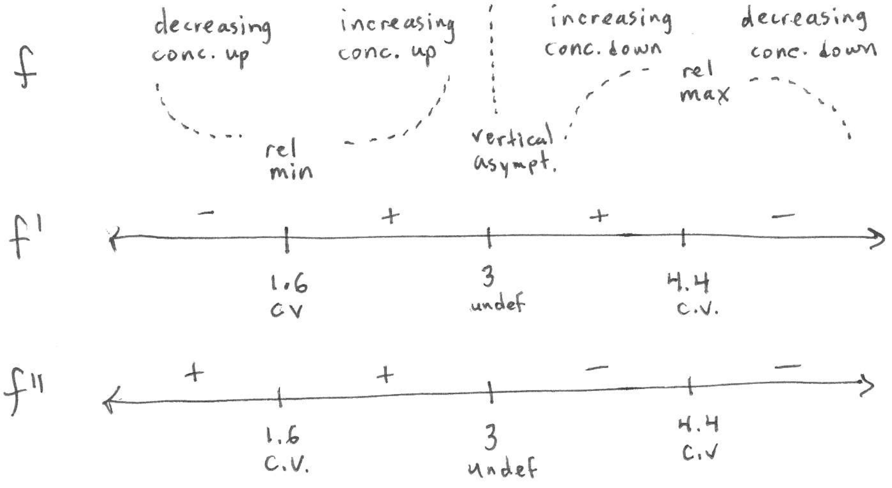
$= \frac{(x-3)\left[(x-3)(-4x+12) - 2(-2x^2+12x-14)\right]}{(x-3)^4}$

$= \frac{\cancel{-4x^2} + \cancel{12x} + \cancel{12x} - 36 + \cancel{4x^2} - \cancel{24x} + 28}{(x-3)^3}$

$= \frac{-8}{(x-3)^3}$

Math 250 Graphing Functions

Combining sign chart info:



$1.6 \approx 3-\sqrt{2}$
$4.4 \approx 3+\sqrt{2}$

rel min at $x = 3-\sqrt{2} \approx 1.6$

$$f(3-\sqrt{2}) = -2(3-\sqrt{2})-5-\frac{4}{(3-\sqrt{2}-3)}$$

$$= -6+2\sqrt{2}-5-\frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$= -11+2\sqrt{2}+2\sqrt{2}$$

$$= -11+4\sqrt{2} \approx -5.3$$

rel. max at $x = 3+\sqrt{2} \approx 4.4$

$$f(3+\sqrt{2}) = -2(3+\sqrt{2})-5-\frac{4}{3+\sqrt{2}-3}$$

$$= -6-2\sqrt{2}-5-\frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$= -11-2\sqrt{2}-2\sqrt{2}$$

$$= -11-4\sqrt{2} \approx -16.7$$

We can now identify the range:

$$(-\infty, -11-4\sqrt{2}) \cup (-11+4\sqrt{2}, \infty)$$

Math 250

Does f(x) cross its oblique asymptote?

set f(x) = eqn of asymptote and solve

$$\frac{-2x^2+x+11}{(x-3)} = -2x-5$$

$$-2x^2+x+11 = (-2x-5)(x-3)$$

$$-\cancel{2x^2}+\cancel{x}+11 = -\cancel{2x^2}+\cancel{6x}\cancel{-5x}+15$$

$11 \neq 15$ no solution.

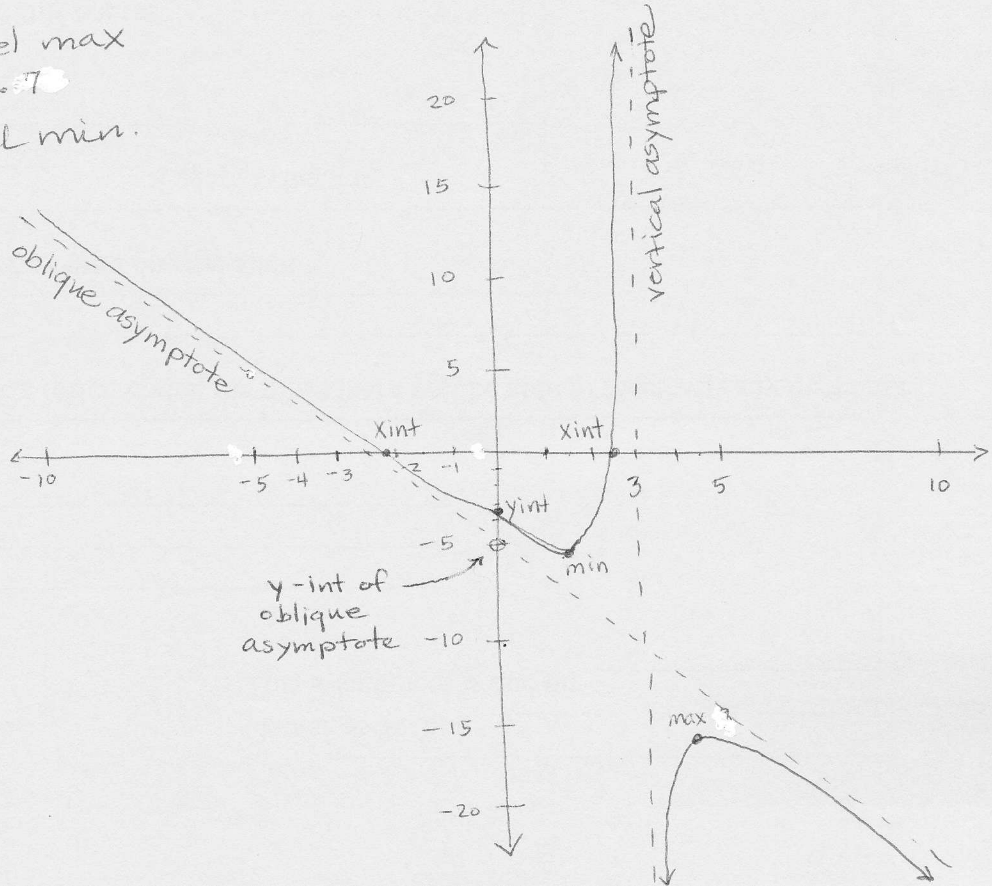
It does not cross the asymptote.

Table of points:

x	y
3	undef vert. asymp.
0	$\frac{-11}{3} \approx -3.7$ y-int
$2.6 \approx -\frac{1}{4}+\frac{\sqrt{89}}{4}$	0 x int
$-2.1 \approx \frac{1}{4}-\frac{\sqrt{89}}{4}$	0 x int
$4.4 \approx 3+\sqrt{2}$	$- - \sqrt{2}$ rel max ≈ -16.7
$1.6 \approx 3-\sqrt{2}$	$- + \sqrt{2}$ rel min. ≈ -5.3

oblique asymptote

$y = -2x-5$



Math 250

② $y = 3(x-1)^{2/3} - (x-1)^2$

x-int: $0 = 3(x-1)^{2/3} - (x-1)^2$

$0 = (x-1)^{2/3}\left[3-(x-1)^{4/3}\right]$

$(x-1)^{2/3} = 0$

$x = 1$

↑

$3 - (x-1)^{4/3} = 0$

$3 = (x-1)^{4/3}$

$3 = \sqrt[3]{(x-1)^4}$

$(3)^3 = (x-1)^4$

$27 = (x-1)^4$

$\pm\sqrt[4]{27} = x-1$

$1 \pm \sqrt[4]{27} = x$

x-ints

$(1,0)$

$(1+\sqrt[4]{27},0)$

$(1-\sqrt[4]{27},0)$

y-int $y = 3(-1)^{2/3} - (-1)^2 = 3-1=2$

$(0,2)$

domain: all ℝ

range: looks like $(-\infty, 2)$

symmetry: none

no asymptotes, holes

$y' = 3\cdot\frac{2}{3}(x-1)^{-1/3} - 2(x-1)$

$y' = 2(x-1)^{-1/3} - 2(x-1)$

M250 cont

$$y'(x) = 2(x-1)^{-1/3}\left[1-(x-1)^{4/3}\right] = \frac{2\left[1-(x-1)^{4/3}\right]}{(x-1)^{1/3}}$$

$y'(x) = 0$ when

$$2\left[1-(x-1)^{4/3}\right] = 0$$

$$1-(x-1)^{4/3} = 0$$

$$(x-1)^{4/3} = 1$$

$$\left(\sqrt[3]{x-1}\right)^4 = 1 \qquad \text{square root property}$$

$$\left(\sqrt[3]{x-1}\right)^2 = \pm 1 \quad \text{(but actually only +1)}$$

$$\sqrt[3]{x-1} = \pm 1$$

$$\sqrt[3]{x-1} = 1 \quad \text{or} \quad \sqrt[3]{x-1} = -1$$

$$x-1 = 1^3 \qquad x-1 = (-1)^3$$

$$x-1 = 1 \qquad x-1 = -1$$

$$x = 2 \qquad x = 0.$$

$y'(x)$ undef when $(x-1)^{1/3} = 0$

$x = 1$

increasing $(-\infty, 0)$
$(1, 2)$

decreasing $(0, 1)$
$(2, \infty)$

$y(0) = 2$ MAX

$y(2) = 2$ MAX

$y(1) = 0$ MIN

$$y'(x) = 2(x-1)^{-1/3} - 2(x-1)$$

$$y''(x) = -\tfrac{2}{3}(x-1)^{-4/3} - 2$$

$y''(x) = 0 \qquad -\tfrac{2}{3}(x-1)^{-4/3} - 2 = 0$ ÷ −2

$y''(x)$ undef at $x = 1$

$$\frac{1}{3(x-1)^{4/3}} + 1 = 0 \qquad \text{rewrite as pos exp}$$

$$\frac{1}{3(x-1)^{4/3}} = -1$$

$$1 = -3(x-1)^{4/3}$$

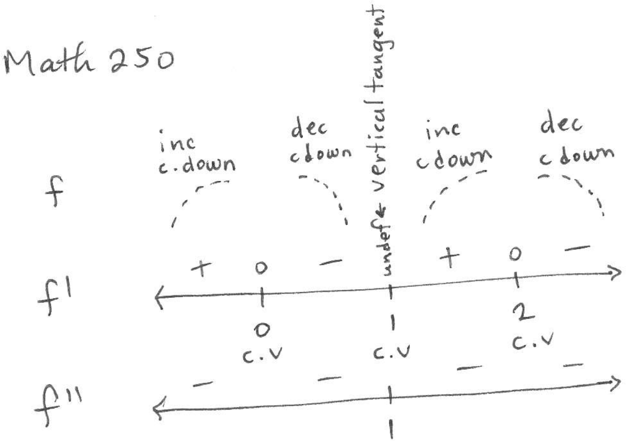
$$-\frac{1}{3} = (x-1)^{4/3}$$

has no solutions → no $\mathbb{R}$ raised to 4th power is negative.

$y''(x)$ is neg everywhere no points of inflection
⇒ graph of y is concave down everywhere.

{This shows (0,2) and (2,2) are max. by 2nd deriv. test.
But 2nd deriv test is inconclusive at (1,0) — must use
1st deriv test to show (1,0) is a rel. min.}

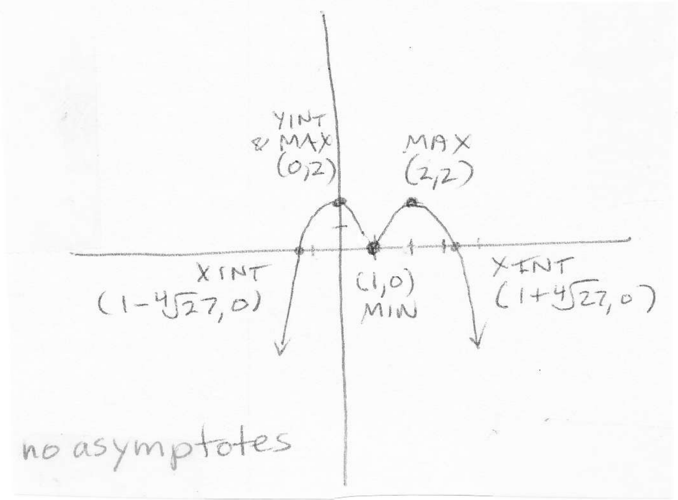
Math 250



f(1) is defined so x=1 is a critical value with critical point (1,0).

$$f(1)=3(1-1)^{2/3}-(1-1)^2$$
$$=3\cdot 0-0$$
$$=0$$

But f'(1) is undefined so the slope of the tangent line to f at x=1 is undefined ⟹ there is a vertical tangent line to f at x=1.



no asymptotes

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③

Recall:

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

if $x = -6 \Rightarrow -(-6) = 6$, $|-6| = 6$

if $x = 6 \Rightarrow 6$, $|6| = 6$

$y = |x^2 - 6x + 5|$

x-int $\quad 0 = |x^2 - 6x + 5|$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 5, 1$

x-ints
(5,0)
(1,0)

y-int $\quad y = |0^2 - 6(0) + 5|$

$y = |5| = 5$

yint
(0,5)

domain: all $\mathbb{R}$
range: $[0, \infty)$
no asymptotes

$$|x^2 - 6x + 5| = \begin{cases} x^2 - 6x + 5 & \text{when } x^2 - 6x + 5 \geq 0 \\ -(x^2 - 6x + 5) & \text{when } x^2 - 6x + 5 < 0 \end{cases}$$

Need to know when $x^2 - 6x + 5 \geq 0$

$(x-5)(x-1) \geq 0$

$$|x^2 - 6x + 5| = \begin{cases} x^2 - 6x + 5 & x \leq 1 \\ -x^2 + 6x - 5 & 1 < x < 5 \\ x^2 - 6x + 5 & x \geq 5 \end{cases}$$

$$y'(x) = \begin{cases} 2x - 6 & x < 1,\ x > 5 \\ -2x + 6 & 1 < x < 5 \end{cases}$$

$y'(1)$ ⟨ $2 - 6 = -4$ one side LEFT
vs
$-2 + 6 = 4$ other side RIGHT

$y'(x)$ is not defined at $x = 1$ and $x = 5$ because left behavior of limit! ≠ right behavior of limit.

$y'(x) = 0$ if $2x - 6 = 0$ has solution in $(-\infty, 1) \cup (5, \infty)$

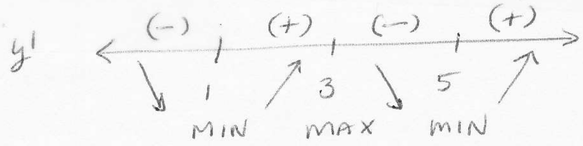
$2x = 6$

$x = 3$ $\quad$ it doesn't.

if $-2x + 6 = 0$ has solution in $(1, 5)$. it does. $x = 3$.

Math 250

critical values $x = 1, 5, 3$



$y'(0) = 2(0) - 6 = -6 < 0$

$y'(2) = -2(2) + 6 = 2 > 0$

$y'(4) = -2(4) + 6 = -2 < 0$

$y'(6) = 2(6) - 6 = 6 > 0$

$y(1) = |1^2 - 6(1) + 5| = 0$ (1,0) min

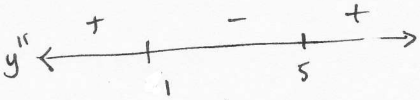
$y(3) = |3^2 - 6(3) + 5| = 4$ (3,4) max

$y(5) = |5^2 - 6(5) + 5| = 0$ (5,0) min.

y decreasing on $(-\infty, 1) \cup (3, 5)$

y increasing on $(1, 3) \cup (5, \infty)$

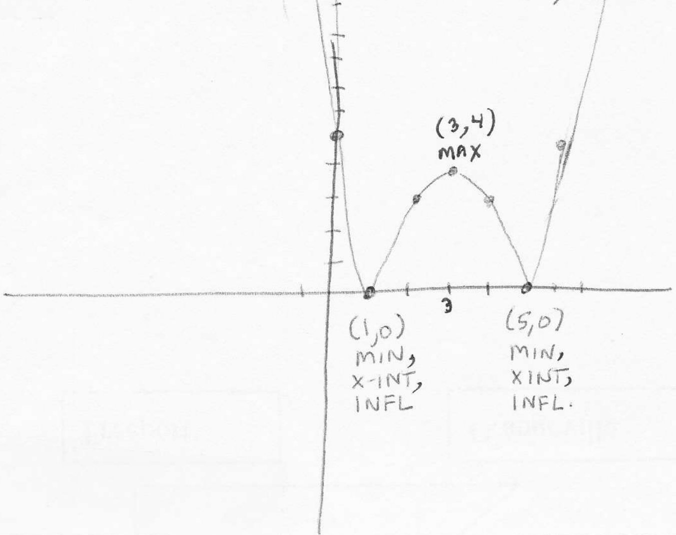
$$y''(x) = \begin{cases} 2 & x < 1,\ x > 5 \\ -2 & 1 < x < 5 \end{cases}$$



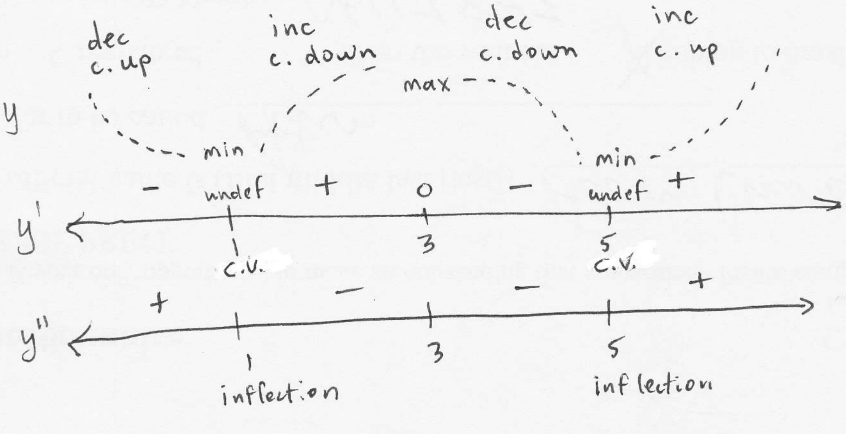
y concave up $(-\infty, 1) \cup (5, \infty)$

y concave down $(1, 5)$

points of inflection (1,0) and (5,0).



x	y
1	0
2	3
4	3
6	5
7	12
−1	12
3	4



1R250

④ $f(x) = x + \frac{32}{x^2} = x \cdot \frac{x^2}{x^2} + \frac{32}{x^2} = \frac{x^3}{x^2} + \frac{32}{x^2} = \frac{x^3+32}{x^2}$

x-intercept(s): $0 = \frac{x^3+32}{x^2}$
set y=0

$0 = x^3 + 32$

$x^3 = -32 \Rightarrow x = -\sqrt[3]{32} = -2\sqrt[3]{4} \approx -3.17$ $\underline{(-\sqrt[3]{32}, 0)}$

y-intercept: $f(0) = \frac{0+32}{0}$ undef.
set x=0

domain: $x \neq 0$ $(-\infty, 0) \cup (0, \infty)$

range: all $\mathbb{R}$ $(-\infty, \infty)$

symmetry: x-axis (no - it's a function) $(x, -y)$

y-axis } $(-x, y)$: $f(-x) = \frac{(-x)^3+32}{(-x)^2} = \frac{-x^3+32}{x^2} = -\left(\frac{x^3-32}{x^2}\right)$
origin }

no.

vertical asymptote: $\underline{x=0}$.

holes: none (no factors cancel)

slant/horizontal asymptotes:

which? $\lim_{x \to \infty} \frac{x^3+32}{x^2} = \infty$

∴ not horizontal asymptote, must be slant:

$$\begin{array}{r} x + 0 \\ x^2 \overline{)\, x^3 + 0x^2 + 0x + 32} \\ \underline{-x^3} \\ 0 \quad 0 \quad\quad 0 + 32 \end{array}$$

$f(x) = \frac{x^3+32}{x} = x + \frac{32}{x^2}$ ⟹ slant asymptote is $\underline{y = x}$

$f'(x) = \frac{x^2(3x^2) - (x^3+32)\cdot(2x)}{x^4}$ or $f'(x) = 1 + \frac{x^2\cdot(0) - 32(2x)}{x^4}$

$= \frac{3x^4 - 2x^4 - 64x}{x^4}$ $= 1 + \frac{-64x}{x^4}$

$= \frac{x^4 - 64x}{x^4}$ $= 1 - \frac{64}{x^3}$

$= \frac{x(x^3-64)}{x^4}$ or $f'(x) = 1 - 64x^{-3}$ (using $f(x) = x + 32x^{-2}$)

$= \frac{x^3 - 64}{x^3}$

$f'(x) = 0$ when $x^3 - 64 = 0$

$x = \sqrt[3]{64} = 4$ $f(4) = 6$ $\underline{(4, 6)}$

$f'(x)$ undef when $x = 0$.

$$f''(x) = \frac{x^3(3x^2)-(x^3-64)(3x^2)}{x^6}$$

OR

$$f'(x) = 1-64x^{-3}$$
$$f''(x) = 192x^{-4}$$
$$= \frac{192}{x^{-4}}$$

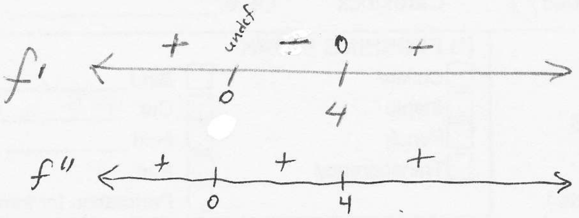
$$= \frac{3x^5-3x^5+192x^2}{x^6}$$

$$= \frac{192x^2}{x^6}$$

$$= \frac{192}{x^4}$$

f'' $\longleftrightarrow$ + concave up always.

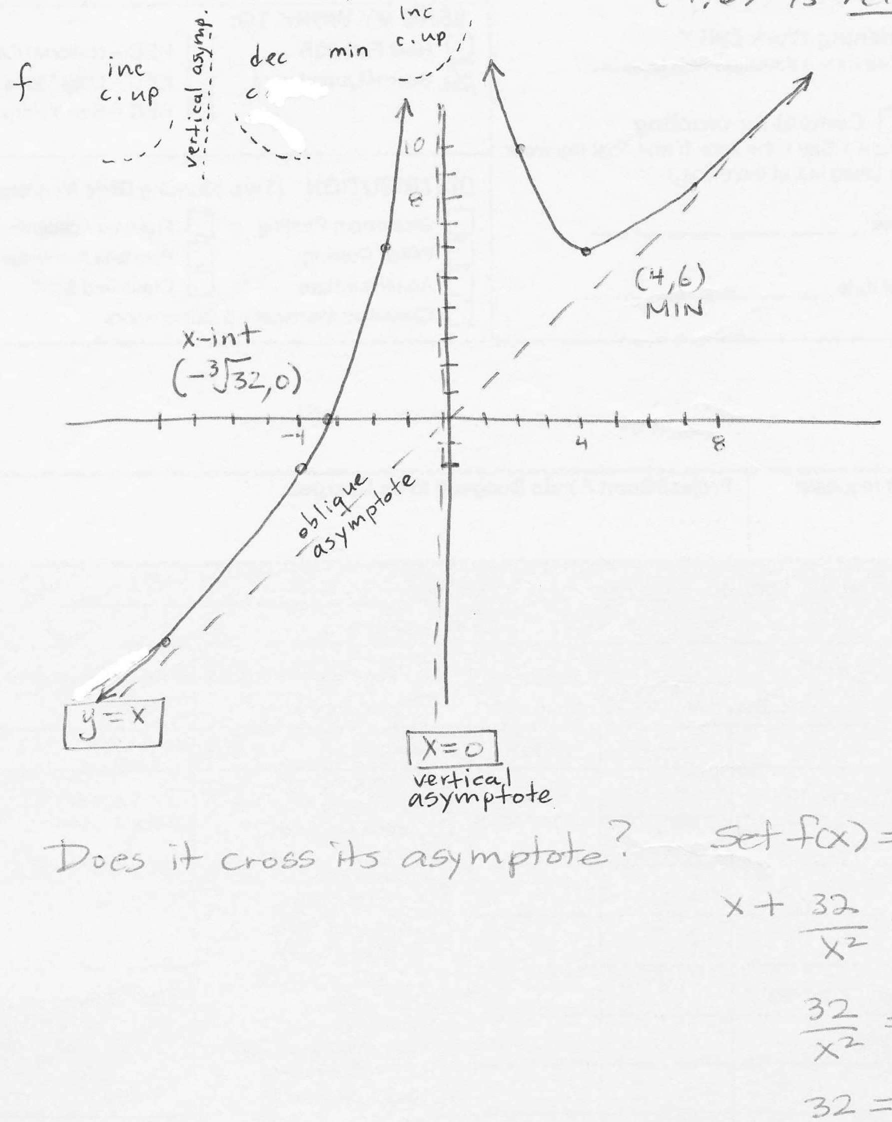
$f''(x)=0$ no where
$f''(x)$ undef $x=0$. } no inflection pts



increasing $(-\infty, 0)$
decreasing $(0, 4)$
increasing $(4, \infty)$

$(4,6)$ is relative min by 1st deriv test

or $f''(4)>0$ by 2nd deriv test



x	f(x)
4	6
8	8.5
2	10
1	33
-1	31
-2	6
-4	-2
-8	-7.5

Does it cross its asymptote?

Set f(x) = equation of asymptote

$$x + \frac{32}{x^2} = x$$

$$\frac{32}{x^2} = 0$$

$$32 = 0$$

no solution

No, it does not.

Math 250

⑤ $g(x) = x\sqrt{9-x} = x(9-x)^{1/2}$

x int $0 = x\sqrt{9-x}$ when $x=0$, $x=9$ $\underline{(0,0)}$ $\underline{(9,0)}$

y int

domain $x \le 9$ $(-\infty, 9]$

range: ~~less obvious~~ — y values are less than a relative max near $x=6$.

Symmetry: none

Vertical asymptotes: none

horizontal / slant: none

holes: none

continuity: continuous everywhere in domain.

$$g'(x) = x \cdot \tfrac{1}{2}(9-x)^{-1/2}(-1) + (9-x)^{1/2} \cdot 1$$ product rule

$$= \frac{-x}{2}(9-x)^{-1/2} + (9-x)^{1/2}$$ tidy up

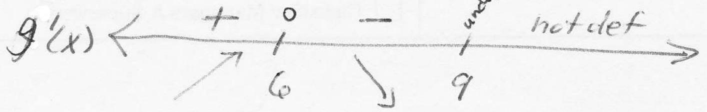
$$= (9-x)^{-1/2}\left[\frac{-x}{2} + (9-x)\right]$$ factor out least powers

$$= (9-x)^{-1/2}\left[-\frac{3}{2}x + 9\right]$$ combine like terms

$$= \frac{-3}{2}\,\frac{(x-6)}{(9-x)^{1/2}}$$ factor out $-\frac{3}{2}$

$g'(x) = 0$ when $x = 6$.

$g'(x)$ undef when $x = 9$.



f increases $(-\infty, 6)$

f decreases $(6, 9)$

$g(6) = 6\sqrt{9-6} = 6\sqrt{3}$ $\underline{(6, 6\sqrt{3})}$ relative max $6\sqrt{3} \approx 10.4$

$$g''(x) = \frac{-3}{2}\left[\frac{(9-x)^{1/2}\,1 - (x-6)\tfrac{1}{2}(9-x)^{-1/2}(-1)}{9-x}\right]$$ quotient rule

$$= \frac{-3}{2}\left[\frac{(9-x)^{1/2} + \tfrac{1}{2}(x-6)(9-x)^{-1/2}}{9-x}\right]$$ tidy up

$$= \frac{-3}{2}(9-x)^{-1/2}\left[\frac{9-x+\tfrac{1}{2}(x-6)}{9-x}\right]$$ factor out least power

$$= \frac{-3}{2}(9-x)^{-1/2}\left[\frac{9-x+\tfrac{1}{2}x-3}{9-x}\right]$$ dist

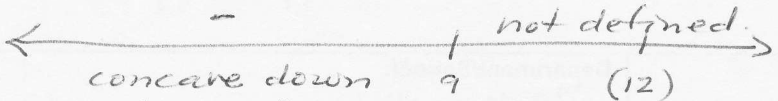
$$= \frac{-3}{2}\left[\frac{6-\tfrac{1}{2}x}{(9-x)^{3/2}}\right]$$ {put $(9-x)^{-1/2}$ below, combine like terms

$$= \frac{+3}{4}\left[\frac{x-12}{(9-x)^{3/2}}\right]$$ factor out $-\frac{1}{2}$.

Math 250

$g''(x) = 0$ when $x = 12$ (not in domain)

$g''(x)$ undef when $x = 9$.

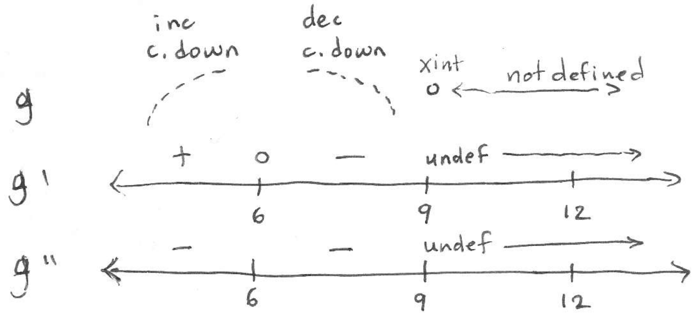


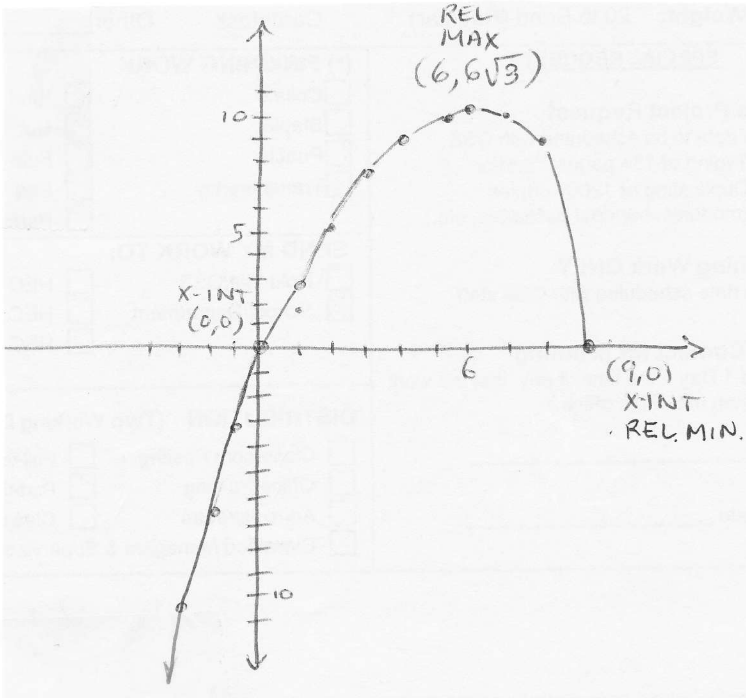
concave down

$(-\infty, 9]$

no points of inflection.

$\lim_{x\to-\infty} \underbrace{x}_{(-)}\underbrace{\sqrt{9-x}}_{(+)} = -\infty$. unbounded downward as $x \to -\infty$.





x	g(x)
−1	−3.2
−2	−6.6
−3	−10.4
−4	−14.4
1	2.8
2	5.3
3	7.3
4	8.9
5	10
7	9.9
8	8
9	

[redacted]

⑥ Analyze and sketch graph

$f(x) = -x + 2\cos x \qquad 0 \le x \le 2\pi$

x-int $\quad 0 = -x + 2\cos x$

$x = 2\cos x$ → approximate values using GC only

x int $\approx (1.0, 0)$

y-int $\quad f(0) = -0 + 2\cos(0)$

$= 2$

$\underline{(0,2)}$ y int

symmetry $\quad f(-x) = -(-x) + 2\cos(-x)$

$= x + 2\cos x$ $\quad$ no.

domain: all $\mathbb{R}$

range: all $\mathbb{R}$

no asymptotes, holes.

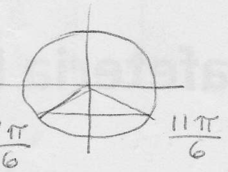
$f'(x) = -1 - 2\sin x$

$f'(x) = 0$

$-1 - 2\sin x = 0$

$-2\sin x = 1$

$\sin x = -\frac{1}{2}$



$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$f'(x)$ defined everywhere

f' ← (−) (+) (−) → $\quad 0 \searrow \frac{7\pi}{6} \nearrow \frac{11\pi}{6} \searrow 2\pi$

MIN $\quad$ MAX

f increasing $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$

f decreasing $\left(0, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$

relative min

$f\left(\frac{7\pi}{6}\right) = -\frac{7\pi}{6} + 2\cos\left(\frac{7\pi}{6}\right)$

$= -\frac{7\pi}{6} + 2\left(\frac{-\sqrt{3}}{2}\right)$

$= -\frac{7\pi}{6} - \sqrt{3}$

relative min

$\left(\frac{7\pi}{6}, -\frac{7\pi}{6} - \sqrt{3}\right) \approx \left(\frac{7\pi}{6}, -5.4\right)$

relative max

$f\left(\frac{11\pi}{6}\right) = -\frac{11\pi}{6} + 2\cos\left(\frac{11\pi}{6}\right)$

$= -\frac{11\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{11\pi}{6} + \sqrt{3}$

relative max

$\left(\frac{11\pi}{6}, -\frac{11\pi}{6} + \sqrt{3}\right) \approx \left(\frac{11\pi}{6}, -4.0\right)$

$f''(x) = -2\cos x$

$f''(x) = 0 \quad -2\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

f'' ← (−) (+) (−) → $\quad 0 \quad \frac{\pi}{2} \quad \frac{3\pi}{2} \quad 2\pi$

f concave up $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

f concave down $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

inflection pts

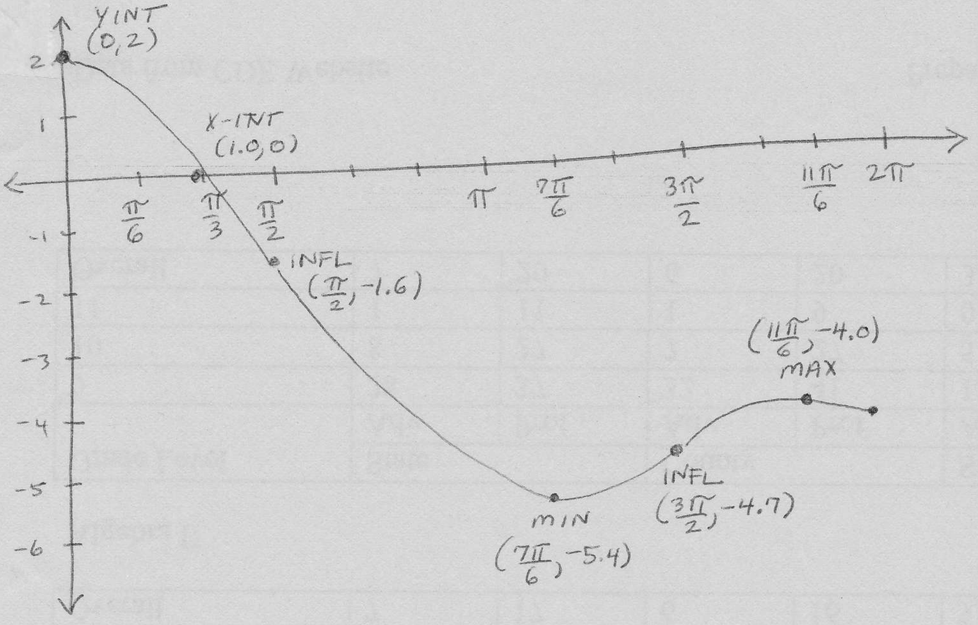
$\left(\frac{\pi}{2}, -\frac{\pi}{2}\right) \quad \left(\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$

$\approx \left(\frac{\pi}{2}, -1.6\right) \approx \left(\frac{3\pi}{2}, -4.7\right)$

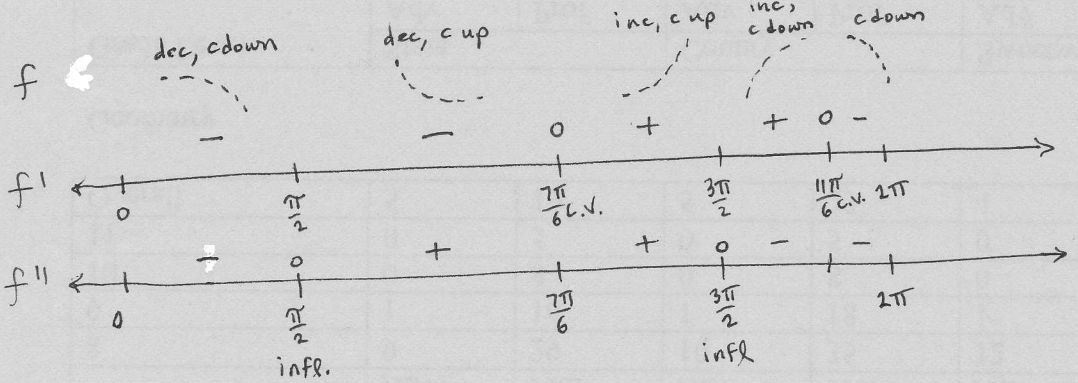
$f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2\cos\frac{\pi}{2} = -\frac{\pi}{2}$

$f\left(\frac{3\pi}{2}\right) = -\frac{3\pi}{2} + 2\cos\left(\frac{3\pi}{2}\right) = -\frac{3\pi}{2}$

17.250 [redacted] [redacted] [redacted]



$$f(2\pi) = -2\pi + 2\cos(2\pi)$$
$$= -2\pi + 2$$
$$\approx -4.3$$



Multiplicity of Roots of Polynomials

$$p(x) = (x-r_1)(x-r_2)^2(x-r_3)^3(x-r_4)^4$$

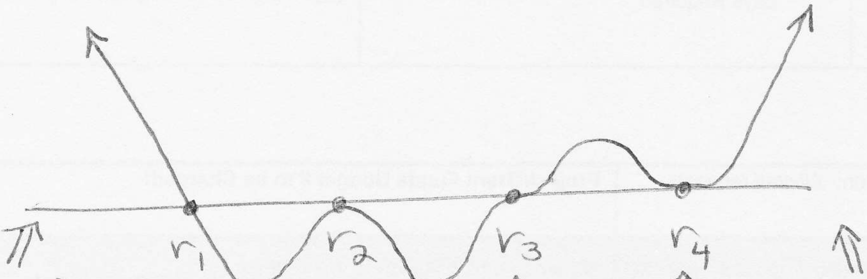
- polynomial must be completely factored to determine multiplicities

$x=r_1$ is a root (or zero)
and its multiplicity is 1
because the exponent on its factor is 1.

$x=r_2$ is a root with multiplicity 2
$x=r_3$ is a root with multiplicity 3
$x=r_4$ is a root with multiplicity 4

- $p(r_1)=0$
  $p(r_2)=0$
  $p(r_3)=0$
  $p(r_4)=0$
  } at each root, y coordinate is zero so each root is an x-intercept

- $p(x)$ is degree 10 (add all multiplicities $1+2+3+4=10$)



⇑ $\lim_{x\to-\infty} p(x) = +\infty$ b/c deg 10

⇑ r_1 is a simple root. Odd mult. Cuts straight through

⇑ r_2 has even mult, so it touches and bounces back, flattening slightly

⇑ r_3 has odd mult, so it passes through, flattening more than mult 2.

⇑ r_4 has even mult, So it bounces back, flattening even more than mult 3.

⇑ $\lim_{x\to\infty} p(x) = +\infty$ b/c deg 10.

small distance from r_4 say $\frac{1}{2}$ becomes $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, very close to x-axis.

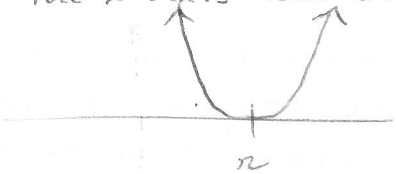
Recall from Math 101 or 244:

Multiplicity of a root of a polynomial function:

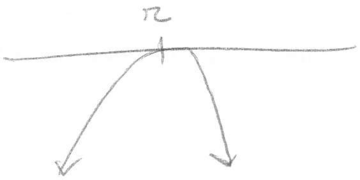
A root $x=r$ of a polynomial $p(x)$ has $p(r)=0$ and multiplicity m if $\frac{p(x)}{(x-r)^m}$ divides evenly but $\frac{p(x)}{(x-r)^{m+1}}$ does not.

i.e. the factor $(x-r)$ appears exactly m times in the fully-factored form of the polynomial $p(x)$.

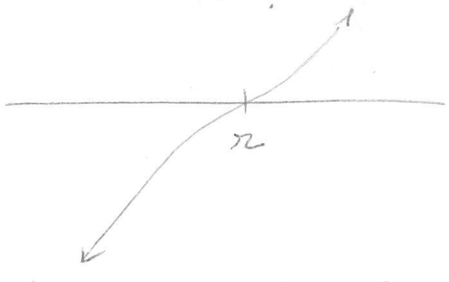
If the multiplicity m is <u>even</u>; the graph at $x=r$ touches the x-axis but does not cross it.



or



If the multiplicity m is <u>odd</u>; the graph at $x=r$ crosses through the x-axis.



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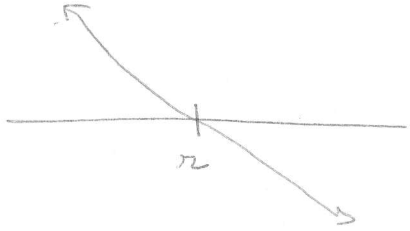


The higher m is, the flatter the graph at $x=r$.

Analyze the polynomial and sketch graph.

⑦ $f(x) = (x-2)^3(x-1)$

$f'(x) = (x-2)^3 \cdot 1 + 3(x-2)^2(x-1)$

$= (x-2)^2[x-2+3(x-1)]$

$= (x-2)^2(x-2+3x-3)$

$= (x-2)^2(4x-5)$

$f''(x) = (x-2)^2 \cdot 4 + (4x-5)(2)(x-2)$

$= 2(x-2)[2(x-2)+4x-5]$

$= 2(x-2)[2x-4+4x-5]$

$= 2(x-2)(6x-9)$

$= 6(x-2)(2x-3)$

x-ints: $f(x) = 0$

$x=2$ is a root of multiplicity 3

$x=1$ is a root of multiplicity 1

y-int: $f(0) = (0-2)^3(0-1)$

$= (-2)^3(-1)$

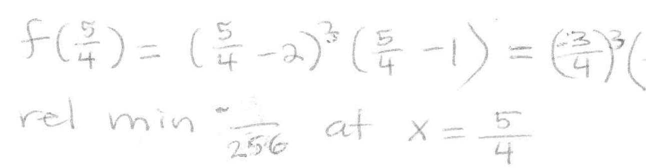
$= 8$

critical values (stationary points) $f'(x) = 0$

$x = 2 \quad x = \frac{5}{4}$

$f''(2) = 0$ inconclusive

1st derivative test



decreasing $(-\infty, \frac{5}{4}]$

increasing $[\frac{5}{4}, \infty)$

$f(\frac{5}{4}) = (\frac{5}{4}-2)^3(\frac{5}{4}-1) = (\frac{-3}{4})^3(\frac{1}{4}) = \frac{-27}{256}$

rel min $\frac{-27}{256}$ at $x = \frac{5}{4}$

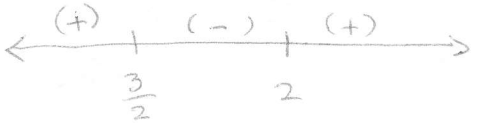
$f''(x) = 0$ at $x = 2, \ x = \frac{3}{2}$

cont →

M250

⑦ cont

f''



concave up $(-\infty, \frac{3}{2}) \cup (2, \infty)$

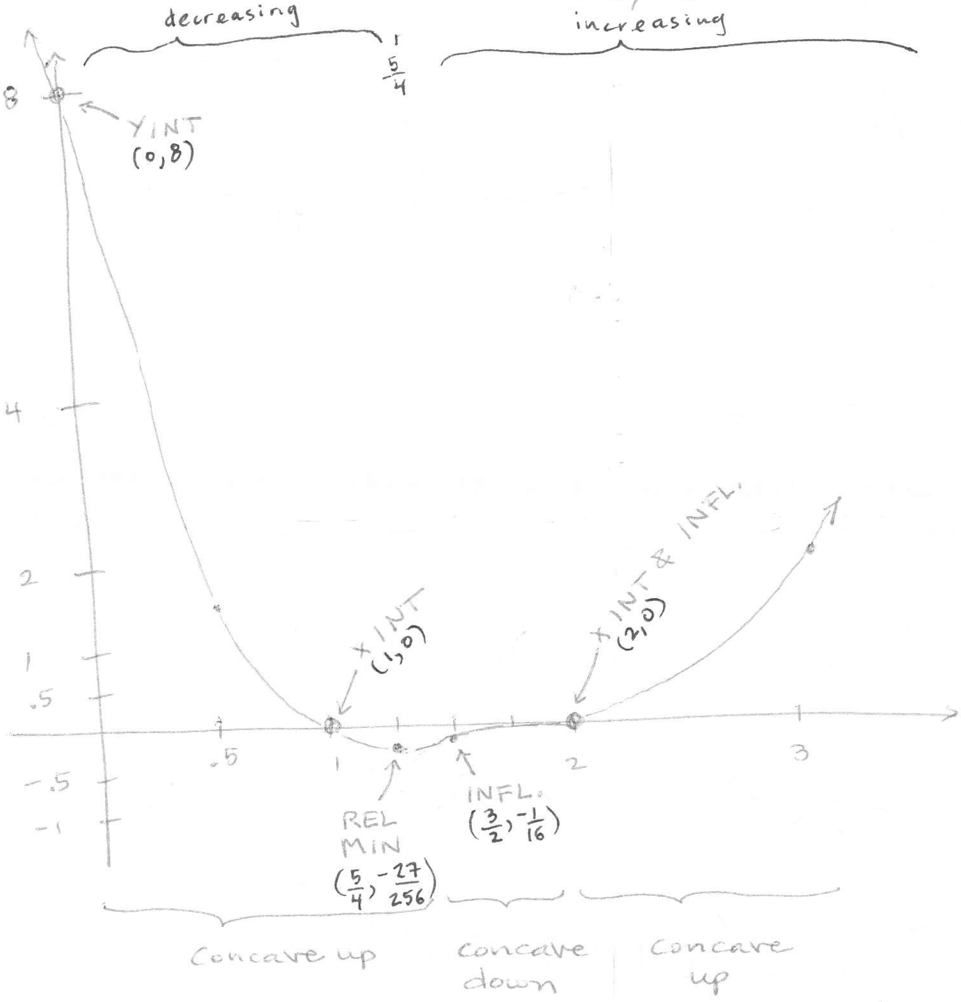
concave down $(\frac{3}{2}, 2)$

both are inflection points

$$f(\tfrac{3}{2}) = (\tfrac{3}{2}-2)^3(\tfrac{3}{2}-1) = (-\tfrac{1}{2})^3(\tfrac{1}{2}) = \tfrac{-1}{16}$$

$$f(2) = (2-2)^3(2-1) = 0$$

inflection points $(\frac{3}{2}, \frac{-1}{16})$ and $(2, 0)$



xints (2,0) m3

(1,0) m1

yint (0,8)

rel min $(\frac{5}{4}, \frac{-27}{256})$

C.V. (2,0)

infl. $(\frac{3}{2}, -\frac{1}{16})$

infl. (2,0)

$\frac{-27}{256} \approx -.11$

$\frac{-1}{16} \approx -.06$

Need x from 0 to 2+

Need y from −.2 to 8

f(3) = 2

f(.5) = 1.6875

To check this graph on GC

use WINDOW SETTINGS:

XMIN = −1

XMAX = 4

YMIN = −.5

YMAX = 1

(use a different window or calculation to check y-int)

Math 250

⑧ Analyze the polynomial and sketch the graph.

$y = 3x^4 - 6x^2 + \frac{5}{3}$

x-int $0 = 3x^4 - 6x^2 + \frac{5}{3}$

$0 = 9x^4 - 18x^2 + 5$

$0 = (3x^2 - 5)(3x^2 - 1)$

$3x^2 = 5$ $\quad$ $3x^2 = 1$

$x^2 = \frac{5}{3}$ $\quad$ $x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{5}{3}}$ $\quad$ $x = \pm\sqrt{\frac{1}{3}}$

$x = \pm\frac{\sqrt{15}}{3}$ $\quad$ $x = \pm\frac{\sqrt{3}}{3}$

xint $\approx (\pm 1.3, 0)$ $\quad$ $\pm(.6, 0)$

Cont →

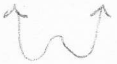
[redacted] Math 250 [redacted]

yint $y = 0 - 0 + \frac{5}{3}$

y int $(0, \frac{5}{3})$

---

domain: all $\mathbb{R}$

range: probably has minimum  quartic shape.

no asymptotes.

Continuous everywhere

symmetric w/RT y-axis

$f(-x) = f(x)$.

$y'(x) = 12x^3 - 12x$

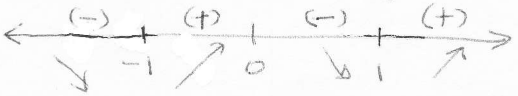
$y'(x) = 0$

$12x^3 - 12x = 0$

$12x(x^2-1) = 0$

$x = 0, 1, -1$

$y'(x)$ defined everywhere



$y'(-2) = 12(-2)((-2)^2-1) < 0$.

$y'(-\frac{1}{2}) = 12(-\frac{1}{2})((-\frac{1}{2})^2-1) > 0$

$y'(\frac{1}{2}) = 12(\frac{1}{2})((\frac{1}{2})^2-1) < 0$

$y'(2) = 12(2)(2^2-1) > 0$

y increasing $(-1, 0) \cup (1, \infty)$

y decreasing $(-\infty, -1) \cup (0, 1)$

$y(1) = 3(1)^4 - 6(1)^2 + \frac{5}{3} = \frac{-4}{3}$

$y(-1) = 3(-1)^4 - 6(-1)^2 + \frac{5}{3} = \frac{-4}{3}$

$(-1, \frac{-4}{3})$ Rel. min

$(1, -\frac{4}{3})$ Rel. min

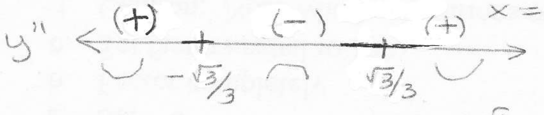
(by 1st deriv. test)

$y''(x) = 36x^2 - 12$

$y''(x) = 0$ $36x^2 - 12 = 0$

$12(3x^2 - 1) = 0$

$x = \pm\frac{\sqrt{3}}{3}$



$y''(-1) = 24 > 0$

$y''(0) = -12 < 0$

$y''(1) = 24 > 0$

{This confirms $(1, -4/3)$ + $(1, 4/3)$ rel min by 2nd deriv. test}

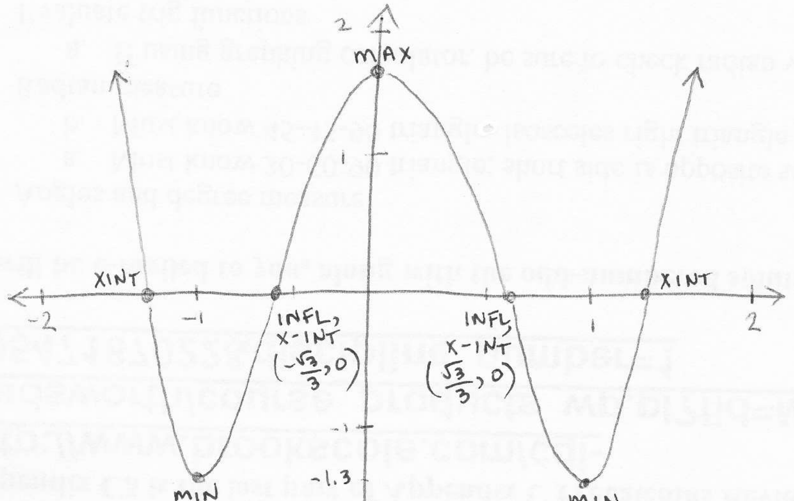
y concave up $(-\infty, \frac{-\sqrt{3}}{3}) \cup (\sqrt{3}/3, \infty)$

y concave down $(\frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$

inflection pts $(\frac{\sqrt{3}}{3}, 0)$

$(-\frac{\sqrt{3}}{3}, 0)$

$\frac{\sqrt{3}}{3} \approx .6$

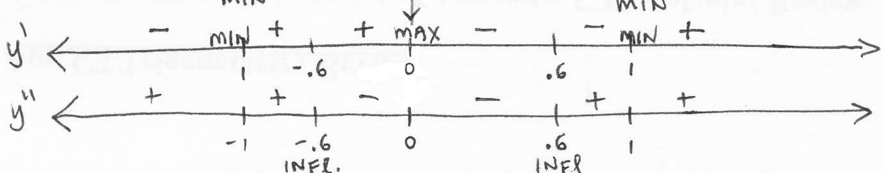


x-ints $(\pm 1.3, 0)$

x-int & infl $(\pm .6, 0)$

y int $(0, \frac{5}{3})$

rel min $(1, -4/3)$ $(-1, -4/3)$



x	y
2	25.7
1	−1.3
−1	−1.3
−2	25.7

Math 250

⑨ $p(x) = (x+2)^2(x-1)$

$$p'(x) = (x+2)^2 \cdot 1 + (x-1) \cdot 2(x+1) \cdot 1$$
$$= (x+2)^2 + 2(x+1)(x-1)$$
$$= (x+2)\,[\,x+2 + 2x-2\,]$$
$$= (x+2)(3x)$$
$$= 3x(x+2)$$
$$= 3x^2 + 6x$$
$$p''(x) = 6x+6$$

intercepts

$p(0) = (0+2)^2(0-1) = -4$ (0, -4) y int

$0 = (x+2)^2(x-1)$

$x=-2 \quad x=1$

(-2, 0) x ints
(1, 0)

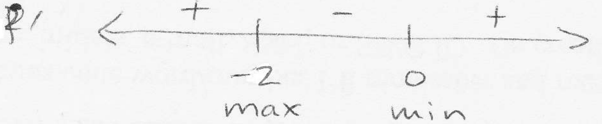
critical values

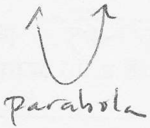
$p'(x) = 0$

$3x(x+2) = 0$

$x = 0, -2$

increasing / decreasing / relative extrema





increasing $(-\infty, -2]$
$[0, \infty)$

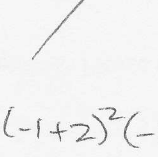
decreasing $[-2, 0]$

$P(-2) = 0$

$P(0) = -4$

rel max (-2, 0)
rel min (0, -4)

concavity / inflection



concave down $(-\infty, -1)$
concave up $(-1, \infty)$
inflection $(-1, -2)$

$P''(x) = 0$

$6x+6=0$

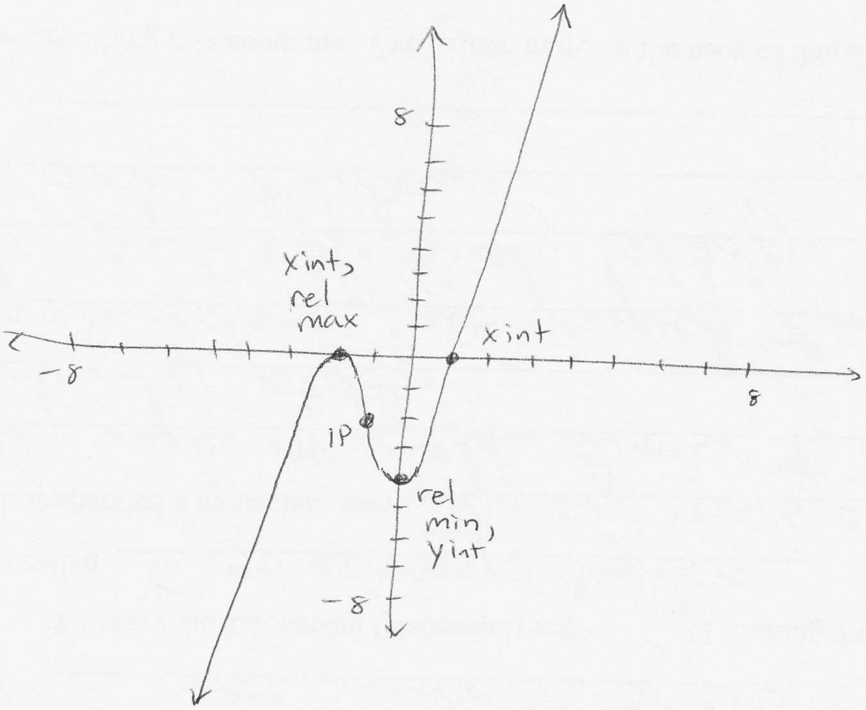
$6x = -6$

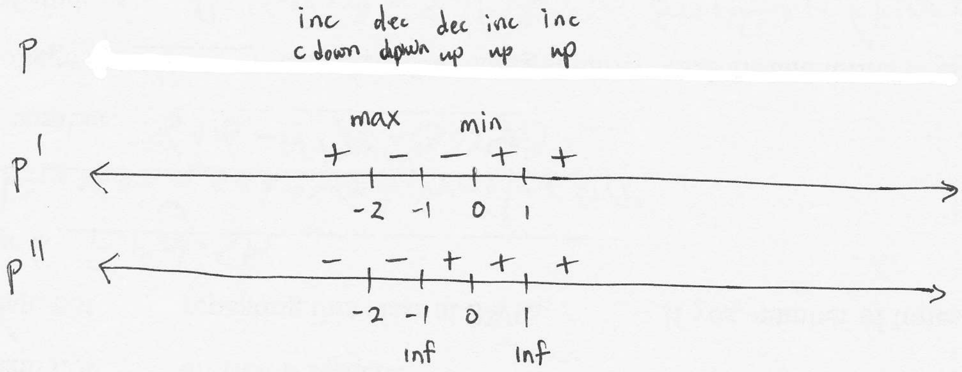
$x = -1$

$P(-1) = (-1+2)^2(-1-1)$
$= 1(-2) = -2$

Math 250

Graph





Math 250

⑩ $p(x) = (x-1)^3(x+2)^2$

$$p'(x) = (x-1)^3 \cdot 2(x+2)\cdot 1 + 3(x-1)^2 \cdot 1 \cdot (x+2)^2$$

$$= 2(x-1)^3(x+2) + 3(x-1)^2(x+2)^2$$

$$= (x-1)^2(x+2)\left[2(x-1) + 3(x+2)\right]$$

$$= (x-1)^2(x+2)\left[2x - 2 + 3x + 6\right]$$

$$= (x-1)^2(x+2)(5x+4)$$

$p''(x)$ 3-way product rule

$$= 2(x-1)(x+2)(5x+4) + (x-1)^2 \cdot 1(5x+4) + (x-1)^2(x+2)\cdot 5$$

$$= 2(x-1)(x+2)(5x+4) + (x-1)^2(5x+4) + 5(x-1)^2(x+2)$$

$$= (x-1)\left[2(x+2)(5x+4) + (x-1)(5x+4) + 5(x-1)(x+2)\right]$$

$$= (x-1)\left[2(5x^2+14x+8) + (5x^2-x-4) + 5(x^2+x-2)\right]$$

$$= (x-1)\left[10x^2+28x+16+5x^2-x-4+5x^2+5x-10\right]$$

$$= (x-1)\left[20x^2+32x+2\right]$$

$$= 2(x-1)\left[10x^2+16x+1\right]$$

b^2-4ac

$= 16^2 - 4(10)(1)$

$= 216$ not factorable

has real roots though....

just irrational

intercepts

$p(0) = (-1)^3(2)^2 = -4$

$0 = (x-1)^3(x+2)^2$

$x=1 \quad x=-2$

$(0,-4)$ y int

$(1,0)$

$(-2,0)$ x ints

critical values

$p'(x) = 0$

$(x-1)^2(x+2)(5x+4) = 0$

$x = 1, -2, -4/5$

Math 250

increasing / decreasing / relative extrema

p' ← + | − | + | + →

$\nearrow$ -2 $\searrow$ $-4/5$ $\nearrow$ 1 $\nearrow$

max rel — rel min — ↑ double root

increasing $(-\infty,-2] \cup [-4/5, 1] \cup [1,\infty)$

decreasing $(-2,-4/5)$

$p(-2) = (-2-1)^3(-2+2) = 0$

$p(-4/5) = (-4/5-1)^3(-4/5+2)^2 = \dfrac{-26244}{3125} \approx -8.39808$

relative max $(-2, 0)$

relative min $\left(-\frac{4}{5}, -8.39808\right)$ approx.

$p(1) = (1-1)^3(1+2)^2 = 0$

stationary pt not extremum $(1,0)$

Concavity / inflection

$p''(x) = 0$

$2(x-1)(10x^2+16x+1) = 0$

$\underset{C}{x=1}$ $\quad x = \dfrac{-16 \pm \sqrt{216}}{2(10)}$

$= \dfrac{-16 \pm \sqrt{2^3 \cdot 3^3}}{20}$

$= \dfrac{-16 \pm 6\sqrt{6}}{20}$

$= \dfrac{-4}{5} \pm \dfrac{3}{10}\sqrt{6} \approx -.0652$ ✓ B

-1.5348 ✓ A

check on GC zeroes of $10x^2+16x+1$

p'' ← − A + B − C + →

-1.5348 $\quad -.0652$ $\quad 1$

no repeated roots — signs alternate

Three inflection points:

$p(1) = 0 \Rightarrow (1,0)$

$p\left(\frac{-4}{5} + \frac{3}{10}\sqrt{6}\right) \approx -4.5241 \Rightarrow (-.1, -4.5)$

$p\left(-\frac{4}{5} - \frac{3}{10}\sqrt{6}\right) \approx -3.5241 \Rightarrow (-1.5, -3.5)$

or, written exactly: $(1,0)$

$\left(-\frac{4}{5} + \frac{3\sqrt{6}}{10}, \overbrace{\left(-\frac{9}{5} + \frac{3\sqrt{6}}{10}\right)^2\left(\frac{6}{5} + \frac{3\sqrt{6}}{10}\right)^2}^{\text{Y-coord's}}\right)$

$\left(-\frac{4}{5} - \frac{3\sqrt{6}}{10}, \left(-\frac{9}{5} - \frac{3\sqrt{6}}{10}\right)^2\left(\frac{6}{5} + \frac{3\sqrt{6}}{10}\right)^2\right)$

Concave down $(-\infty, -1.5348) \cup (-.0652, 1)$ approx.

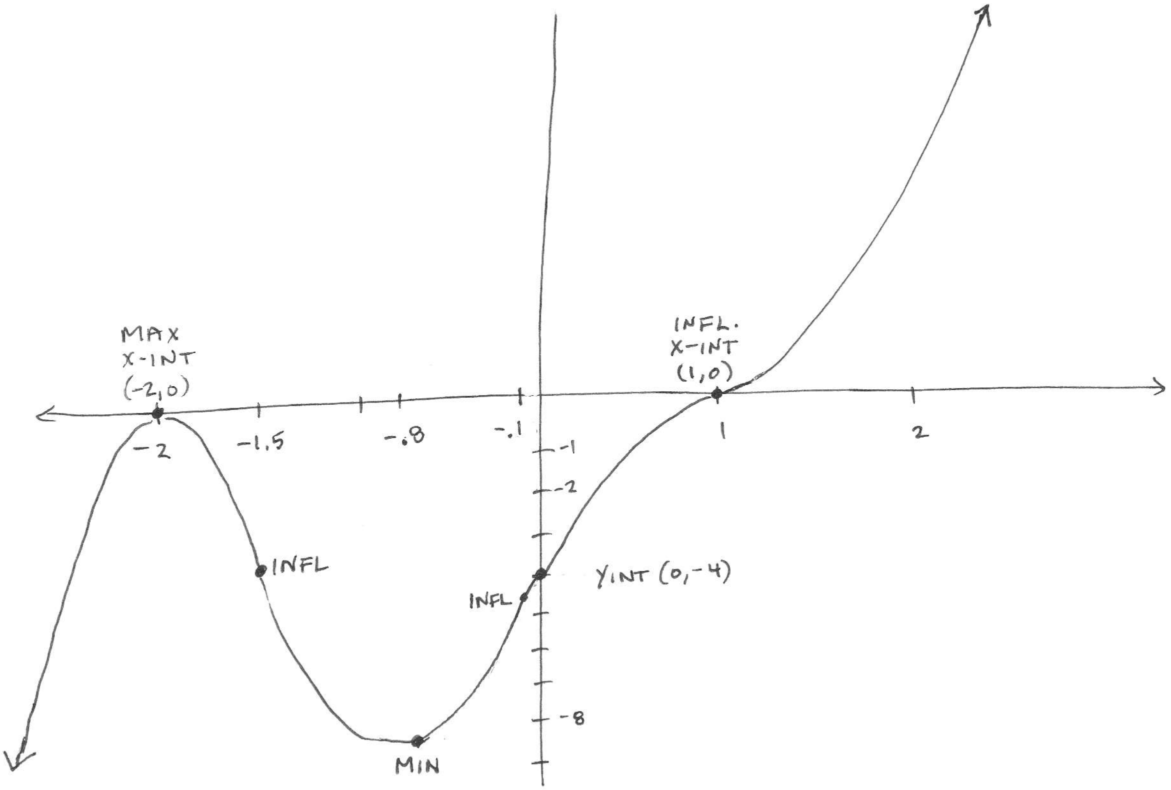
concave up $(-1.5348, -.0652) \cup (1, \infty)$ approx.

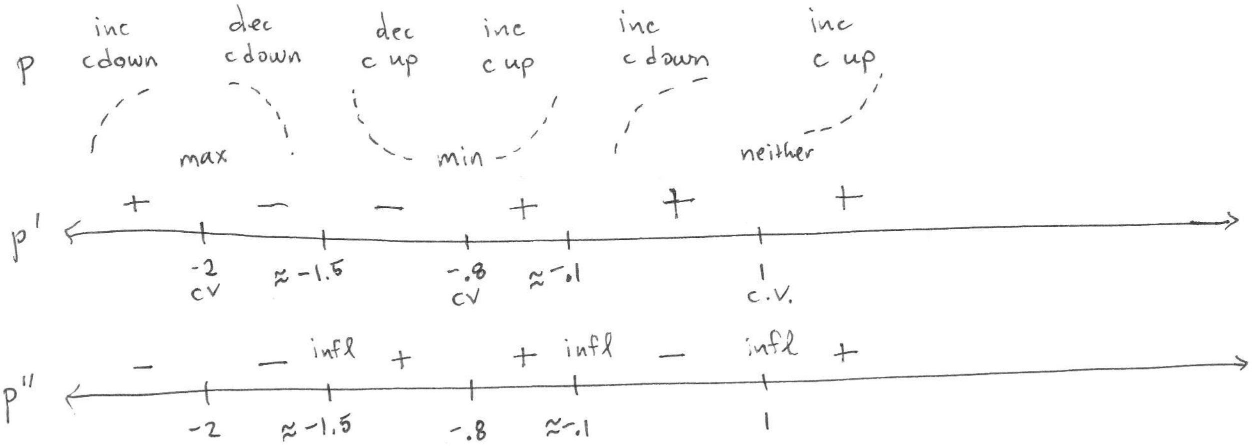
inflection $(1,0)$

approx. $(-.0652, -4.5241)$

approx. $(-1.5348, -3.5241)$

Math 250


MAX
X-INT
(-2,0)
-2
-1.5
-.8
-.1
-1
-2
INFL
INFL
YINT (0,-4)
-8
MIN
INFL.
X-INT
(1,0)
1
2


p
inc
cdown
dec
cdown
dec
c up
inc
c up
inc
c down
inc
c up
max
min
neither
p'
+
−
−
+
+
+
-2
CV
≈ -1.5
-.8
CV
≈ -.1
1
c.v.
p''
−
−
infl
+
+
infl
−
infl
+
-2
≈ -1.5
-.8
≈ -.1
1

⑪ Analyze and sketch.

$$f(x) = \frac{(3x+1)^2}{(x-1)^2} = \frac{9x^2+6x+1}{x^2-2x+1}$$

x ints: $f(x)=0$

$(3x+1)^2=0$

$3x+1=0$

$x=-\frac{1}{3}$

y-ints: $f(0) = \frac{(3\cdot 0+1)^2}{(0-1)^2} = 1$

vertical asymptote: $x-1=0$

$x=1$

no holes.

horizontal asymptote: $y=\frac{9x^2}{x^2}=9$

no slant or curvilinear.

Symmetry: $f(-x) = \frac{(3(-x)+1)^2}{(-x-1)^2}$

$= \frac{(-3x+1)^2}{(-x-1)^2} \neq f(x)$
$\neq -f(x)$

no symmetry on x, y, or origin.

discontinuous at $x=1$.

domain: all $\mathbb{R}$ except $x=1$

range: can $f(x)=9$?

$\frac{(3x+1)^2}{(x-1)^2}=9$

$(3x+1)^2=9(x-1)^2$

$3x+1=\pm 3(x-1)$

$3x+1=3(x-1)$ or $3x+1=-3(x-1)$

$3x+1=3x-3$ $\quad$ $3x+1=-3x+3$

$1\neq -3$ $\quad$ $6x=2$

$x=\frac{1}{3}$.

yes.

The graph crosses the horizontal asymptote at $x=\frac{1}{3}$ $\quad f(\frac{1}{3})=9$ ✓

More about range in max/min.

1 2 3 4 5

$$f(x) = \frac{(3x+1)^2}{(x-1)^2} = \left(\frac{3x+1}{x-1}\right)^2 = (3x+1)^2(x-1)^{-2} = \left(3+\frac{4}{x-1}\right)^2 = (3+4(x-1)^{-1})^2$$

$$\begin{array}{r|rr} 1 & 3 & 1 \\ & & 3 \\ \hline & 3 & 4 \end{array}$$

Method 5: $f'(x) = 2\,(3+4(x-1)^{-1})(-4(x-1)^{-2})$

$$= \frac{-8}{(x-1)^2}\left(3+\frac{4}{x-1}\right)$$

$$= \frac{-8(3x+1)}{(x-1)^3}$$

Method 2: $f'(x) = 2\left(\frac{3x+1}{x-1}\right)^1 \cdot \left[\frac{(x-1)\cdot 3 - (3x+1)\cdot 1}{(x-1)^2}\right]$

$$= \frac{2(3x+1)}{(x-1)^3}[3x-3-3x-1]$$

$$= \frac{2(3x+1)}{(x-1)^3}[-4]$$

$$= \frac{-8(3x+1)}{(x-1)^3}$$

Method 1: $f'(x) = \frac{(x-1)^2\cdot 2(3x+1)\cdot 3 - (3x+1)^2\cdot 2(x-1)\cdot}{(x-1)^4}$

$$= \frac{6(x-1)^2(3x+1) - 2(x-1)(3x+1)^2}{(x-1)^4}$$

$$= \frac{2(x-1)(3x+1)[3(x-1)-(3x+1)]}{(x-1)^4}$$

$$= \frac{2(3x+1)[3x-3-3x-1]}{(x-1)^3}$$

$$= \frac{2(3x+1)(-4)}{(x-1)^3}$$

$$= \frac{-8(3x+1)}{(x-1)^3}$$

Method 3: $f'(x) = (3x+1)^2\cdot(-2)(x-1)^{-3} + (x-1)^{-2}\cdot 2(3x+1)\cdot 3$

$$= -2(x-1)^{-3}(3x+1)^2 + 6(x-1)^{-2}(3x+1)$$

$$= -2(x-1)^{-3}(3x+1)[3x+1-3(x-1)]$$

$$= -2(x-1)^{-3}(3x+1)(3x+1-3x+3)$$

$$= -2(x-1)^{-3}(3x+1)(4)$$

$$= \frac{-8(3x+1)}{(x-1)^{-3}}$$

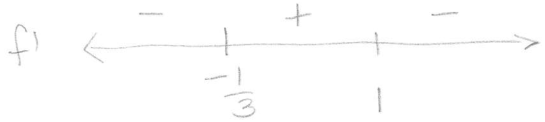
$$f'(x) = \frac{-8(3x+1)}{(x-1)^3}$$

Critical values $f'(x)=0$ when $3x+1=0$

$x=-\frac{1}{3}$ stationary point

$f'(x)$ undef when $x-1=0$

$x=1$



increasing $[-\frac{1}{3}, 1)$ ← exclude 1 because f(1) undefined

decreasing $(-\infty, -\frac{1}{3}] \cup (1, \infty)$

relative minimum at $x=-\frac{1}{3}$
$f(-\frac{1}{3}) = 0$ } also x-int.

$$f'(x) = -8(3x+1)(x-1)^{-3}$$

$$f''(x) = -8\{(3x+1)(-3)(x-1)^{-4} + 3(x-1)^{-3}\}$$

$$= 24(x-1)^{-4}\{(3x+1)-(x-1)\}$$

$$= 24(x-1)^{-4}[3x+1-x+1]$$

$$= 24(x-1)^{-4}[2x+2]$$

$$= \frac{48(x+1)}{(x-1)^4}$$

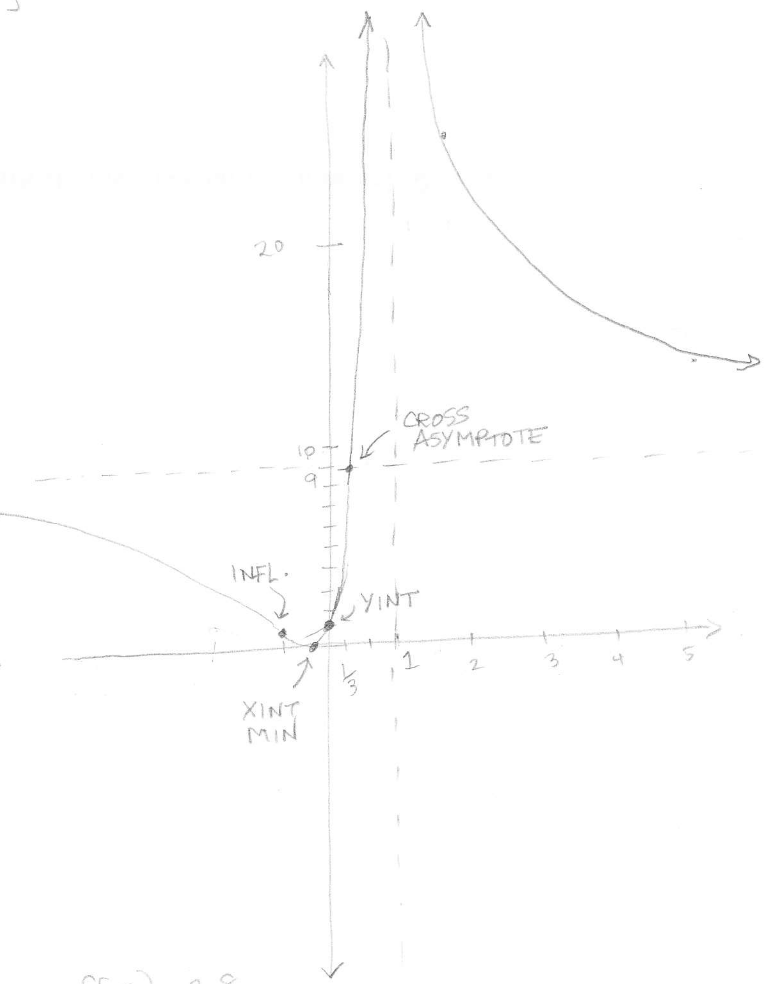
$f''(x)=0$ when $x=-1$

$f''(x)$ undef when $x=1$

concave up $(-1,1) \cup (1,\infty)$

concave down $(-\infty,-1)$

inflection pt $x=-1$ $f(-1)=1$



$f(-2) = 2.8$

$f(\frac{2}{3}) = 81$

$f(2) = 49$

$f(3) = 25$

$f(-5) = 5.4$

⑫

$$f(x) = \frac{(2+3x-x^3)}{x} = 2x^{-1}+3-x^2$$

$$f'(x) = -2x^{-2}-2x = -2x^{-2}(1+x^3) = \frac{-2(1+x^3)}{x^2}$$

$$f''(x) = 4x^{-3}-2 = 2x^{-3}(2-x^3) = \frac{2(2-x^3)}{x^3}$$

x-int: $2+3x-x^3=0 \Rightarrow x^3-3x-2=0$

GC $x=2$?

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -3 & -2 \\ & & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & 0 \end{array} \checkmark$$

$x^2+2x+1=0$

$(x+1)^2$

$$f(x) = \frac{(x-2)(x+1)^2}{x}$$

x-ints $(2,0)$
$(-1,0)$ mult 2

y-int $f(0)$ undef $\Rightarrow$ no y int

$f(-x) = 2(-x)^{-1}+3-(-x)^2$
$= -2x^{-1}+3-x^2$
$\neq -f(x)$
$\neq f(x)$ no symmetry

vertical asymptote $x=0$

no holes, horizontal or slant asymptotes.

curvilinear asymptote: divide, writing numerator in standard form {May need long or synthetic division for polynomial denominators}

$$f(x) = \frac{-x^3}{x}+\frac{3x}{x}+\frac{2}{x}$$
$$= -x^2+3+\frac{2}{x}$$

end behavior as $x \to \infty$ is like $-x^2+3$ because $\lim_{x\to\infty} \frac{2}{x} = 0$

curvilinear asymptote $y=-x^2+3$

Does f(x) cross this asymptote?

$$\frac{2+3x-x^3}{x} = -x^2+3$$

$$\frac{2}{x}+3-x^2 = -x^2+3$$

$$\frac{2}{x} = 0$$

$\to$ $2 \neq 0$
no solution
No, f(x) does not cross
$y=-x^2+3$

f is discontinuous only at $x=0$.
f is differentiable everywhere except $x=0$

$f'(x)=0 \qquad 1+x^3=0$
$x^3=-1$
$x=\sqrt[3]{-1}=-1$ also two complex solutions which do not appear on the graph.

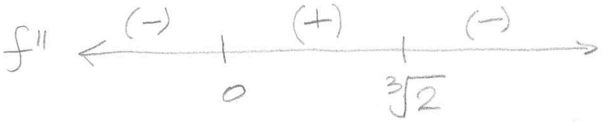
critical value (stationary point) at $x=-1$

$f'(x)$ undefined $x^{-2} \Rightarrow \frac{1}{x^2}$ $x=0$. critical value, but not stationary.

f increasing $(-\infty,-1]$
f decreasing $[-1,0)\cup(0,\infty)$
relative max at $x=-1$ $f(-1)=0$ $\underline{(-1,0)}$

$f''(x)=0 \qquad 2-x^3=0$
$x^3=2$
$x=\sqrt[3]{2}$ also two complex solutions, not on graph

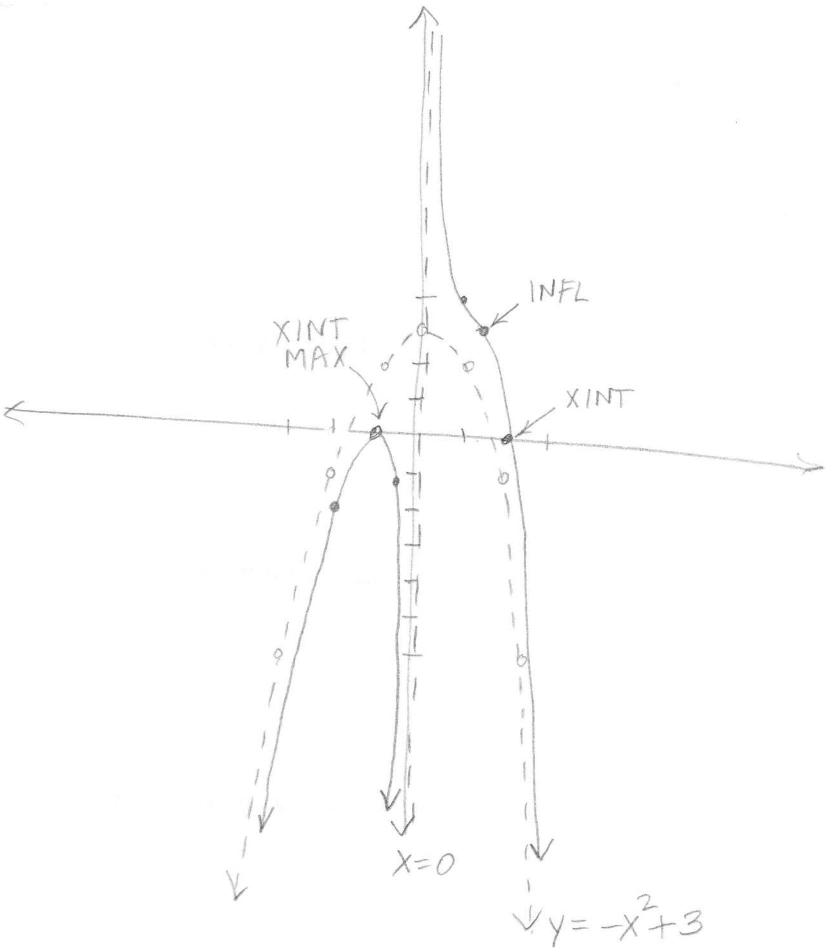
$f''(x)$ undefined $x^{-3}=\frac{1}{x^3} \Rightarrow x=0$



f concave up $(0,\sqrt[3]{2})$
f concave down $(-\infty,0)\cup(\sqrt[3]{2},\infty)$
inflection point $\rightarrow$ not $x=0$ because $f(0)$ undefined

$$x=\sqrt[3]{2}, \quad f(\sqrt[3]{2})=\frac{2}{\sqrt[3]{2}}+3-(\sqrt[3]{2})^2$$
$$=\frac{2\sqrt[3]{4}}{2}+3-\sqrt[3]{4}$$
$$=\sqrt[3]{4}+3-\sqrt[3]{4}$$
$$=3$$

$\underline{(\sqrt[3]{2},3)}$



(2, 0) x-int

(−1, 0) x int
mult 2

no y int

x=0 asymptote

vertex (0,3) $\Leftarrow y=-x^2+3$ asymptote

increase $(-\infty, -1]$

decrease elsewhere

rel max (−1, −6)

inflection $(\sqrt[3]{2}, 3)$

x : $-1 \longleftrightarrow \sqrt[3]{2}$

y : $-6 \longleftrightarrow 3$

X	Y
−2	−2
$-\frac{1}{2}$	−1.25
$\sqrt[3]{2} \approx$ 1.2	3
1	4

Additional worked examples attached to the end of these notes.

Math 250

⑬ Find the slant asymptote of $f(x) = \frac{5x^4 - x^3}{3x^3 + 2x^2 + 1}$

deg(num) − deg(denom) = deg (expect on quotient).

$4 \quad - \quad 3 \quad = \quad 1$ ⇐ we expect a linear result

as $x \to \pm\infty$ graph will approach a line.

Precalculus Method: Long division

$$\begin{array}{r|l} & \frac{5}{3}x - \frac{13}{9} \\ \hline 3x^3+2x^2+1 \,) & 5x^4 - x^3 + 0x^2 + 0x + 0 \\ & 5x^4 + \frac{10}{3}x^3 \qquad\quad + \frac{5}{3}x \\ \hline & -\frac{13}{3}x^3 + 0x^2 - \frac{5}{3}x + 0 \\ & -\frac{13}{3}x^3 - \frac{26}{9}x^2 \qquad - \frac{13}{9} \\ \hline & \frac{26}{9}x^2 - \frac{5}{3}x + \frac{13}{9} \end{array}$$

- * Descending exponents
- * Placeholders 0 for all missing terms
- * line up like terms
- * if you put quotient terms above their like terms, you'll be done when you get to the end of the division symbol

$$f(x) = \frac{5}{3}x - \frac{13}{9} + \frac{\left(\frac{26}{9}x^2 - \frac{5}{3}x + \frac{13}{9}\right)\cdot 9}{(3x^3 + 2x^2 + 1)\cdot 9}$$

$$f(x) = \underbrace{\frac{5}{3}x - \frac{13}{9}}_{\text{asymptote}} + \underbrace{\frac{26x^2 - 15x + 13}{9(3x^3 + 2x^2 + 1)}}_{\text{remainder}}$$

$(3x^3)\cdot(?) = \frac{-13}{3}x^3$

means $(?) = \frac{\frac{-13}{3}}{3} = \frac{-13}{9}$

$\boxed{\text{slant asymptote } y = \frac{5}{3}x - \frac{13}{9}} \Rightarrow 9y = 15x - 13$

$\boxed{15x - 9y - 13 = 0}$ General form.

Does the $\lim_{x \to \infty} f(x)$ tell us the equation of the asymptote in this case? No – we can see the slope $\frac{5}{3}$, but not the y-int $\frac{-13}{9}$ that we need to graph the line exactly.

$$\lim_{x \to \infty} \frac{5x^4 - x^3}{3x^3 + 2x^2 + 1}$$

$$= \lim_{x \to \infty} \left[\frac{\frac{5x^4}{x^3} - \frac{x^3}{x^3}}{\frac{3x^3}{x^3} + \frac{2x^2}{x^3} + \frac{1}{x^3}}\right]$$

$$= \lim_{x \to \infty} \left[\frac{5x - 1}{3 + \frac{2}{x} + \frac{1}{x^3}}\right]$$

$= \boxed{+\infty}$ DNE, unbounded.